

# Bose-Einstein condensation for interacting scalar fields in curved spacetime

Klaus Kirsten [<sup>†</sup>]

Departament d'ECM, Facultat de Física  
Universitat de Barcelona, Av. Diagonal 647, 08028 Barcelona  
Catalonia, Spain

and

David J. Toms [<sup>‡</sup>]

Department of Physics, University of Newcastle Upon Tyne,  
Newcastle Upon Tyne, United Kingdom NE1 7RU

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## Abstract

We consider the model of self-interacting complex scalar fields with a rigid gauge invariance under an arbitrary gauge group  $G$ . In order to analyze the phenomenon of Bose-Einstein condensation finite temperature and the possibility of a finite background charge is included. Different approaches to derive the relevant high-temperature behaviour of the theory are presented.

# 1 Introduction

Although Bose-Einstein condensation is well-known to occur for non-relativistic spin-0 bosons at low temperatures, it is only comparatively recently that the analogous phenomenon was studied in relativistic quantum field theory [1, 2, 3, 4, 5]. For relativistic fields Bose-Einstein condensation occurs at high temperatures and can be interpreted in terms of spontaneous symmetry breaking.

The extension and generalization of Bose-Einstein condensation to curved spacetimes, and to spacetimes with boundaries has also been the subject of much study. The non-relativistic Bose gas in the Einstein static universe was treated by Altaie [6]. The generalization to relativistic scalar fields was given in Refs. [7, 8]. An extension to higher dimensional spheres was given by Shiraishi [9]. More recently a detailed critical examination of Bose-Einstein condensation in the static Einstein universe has been performed [10]. Bose-Einstein condensation on hyperbolic manifolds [11], and in the Taub universe [12] has also been considered.

By calculating the high temperature expansion of the thermodynamic potential when boundaries are present, Ref. [13] was able to examine Bose-Einstein condensation in certain cases. Later work [14] showed how to interpret Bose-Einstein condensation in terms of symmetry breaking in the manner of the flat spacetime calculations [2, 3]. A crucial feature when the spacetime is curved, or when boundaries are present, is that if the symmetry breaks then the ground state cannot be constant in general [14]. A later study [15] showed how interacting scalar fields could be treated.

We want to continue the considerations on the Bose-Einstein condensation taking as a model a set of interacting scalar fields with a rigid gauge invariance under the action of an arbitrary gauge group  $G$ . In order to examine the phenomenon of Bose-Einstein condensation in relativistic quantum field theories, the interesting regime is the high-temperature one to which we will concentrate from the beginning. In doing that, different approaches may be developed and we will present several of them at different stages of the paper.

For explicitity and in order to develop the zeta function approach in combination with heat-kernel techniques, we concentrate first on the  $O(N)$  self-interacting Bose gas in curved spacetime at finite temperature and finite chemical potentials. This continues the analysis started in [1, 2, 3] from flat to curved spacetime. In section 2 we describe in some detail the model and derive briefly the equations necessary for the following considerations. The aim of section 3 is to derive the high-temperature behaviour of the theory using an expansion around the free field theory. The leading dependence on the chemical potentials  $\mu_\alpha$  will be given by the one of the free field. The leading term due to interaction may be determined by considering the theory with  $\mu_\alpha = 0$  (this may be seen by dimensional analysis). For that reason, in section 3 we derive the high-temperature behaviour of the Bose gas for vanishing chemical potential and for arbitrary  $N$ . In that case it is no additional complication to allow for nonconstant background field and we will consider this general case. Afterwards, in section 4, we concentrate on the  $O(2)$  model and we restrict our attention to a constant background field. Because we are only

interested in the leading dependence of the effective action on the temperature, this is actually no restriction because by dimensional grounds it is clear that derivatives of the background field will not enter. This has however the advantage that calculations are much more explicit in that for example an explicit knowledge of the excitation energies is given. We will develop two different approaches. The first one is once more based on the zeta function approach, the second one is based on a weak field expansion of the excitation energies. In both cases we determine the leading two terms in the high temperature expansion. Section 5 contains remarks on the limit  $N \rightarrow \infty$  in the  $O(N)$  model, especially the influence of higher loop corrections is briefly discussed. In section 6 we extend our results to general gauge groups. Here we choose the approach based on the excitation energies. In that part the chemical potential is treated as a small interaction term and we use normal quantum-field perturbation theory. In this approach and for the general gauge group we indicate how results containing higher orders in the temperature expansion may be obtained and how a nonconstant background field may be treated with in a systematic way.

## 2 Incorporation of $\mu$ in an $O(N)$ model of a self-interacting Bose gas

In this section we will derive the basic equations of a model of  $N$  real scalar fields with an arbitrary self-interaction coupled to a classical gravitational field. We are especially interested in the influence that conserved background charges have on the theory. As has been argued in detail [2], the maximum number of mutually commuting charges is  $N/2$  for  $N$  even and for definiteness we will always take  $N$  even.

We will consider the  $O(N)$  model in an arbitrary ultrastatic spacetime [16] with

$$ds^2 = dt^2 - g_{ij}(\mathbf{x})dx^i dx^j . \quad (2.1)$$

$x^i$  are local coordinates on the spatial hypersurface  $\Sigma$  which is taken to be a  $D$ -dimensional Riemannian with a finite volume  $V$ . The infinite volume limit can be taken at the end. The scalar field action is

$$S = \int dt \int_{\Sigma} d\sigma_x \left\{ \frac{1}{2} (\partial^\mu \Phi_a) (\partial_\mu \Phi_a) - \frac{1}{2} (m^2 - \xi R) |\Phi|^2 - V_{tree}(\Phi) \right\}, \quad (2.2)$$

where  $V_{tree}(\Phi)$  is the interaction potential at the tree level. We will assume that this potential only depends on  $\Phi_a$  through  $|\Phi|^2 = \Phi_a \Phi_a$ . Here and in the following the summation over  $a = 1, \dots, N$  is always included.  $d\sigma_x = \sqrt{g_{ij}(\mathbf{x})} d^D x$  is the volume element on  $\Sigma$ .

Associated with the invariance of (2.2) under global gauge transformations, there are  $N(N-1)/2$  conserved currents  $J_{ab}^\mu$  giving rise to the conserved charges  $Q_{ab}$  :

$$Q_{ab} = \int_{\Sigma} d\sigma_x J_{ab}^0 \quad (2.3)$$

where

$$J_{ab}^0 = i(\Phi_a \dot{\Phi}_b - \Phi_b \dot{\Phi}_a) . \quad (2.4)$$

A quantum state of the system is specified by the eigenvalues of a complete set of mutually commuting charges. As mentioned (see [2]), the maximum number of mutually commuting charges is  $N/2$ . A convenient set of mutually commuting charges is  $Q_{12}, Q_{34}, \dots$ , which we label by

$$Q_\alpha = Q_{2\alpha-1, 2\alpha} = \int_\Sigma d\sigma_x j_\alpha^0. \quad (2.5)$$

The conservation of charges is implemented in the usual way by introducing Lagrange multipliers  $\mu_\alpha$  and defining the generalized Hamiltonian density  $\bar{\mathcal{H}}$  by

$$\bar{\mathcal{H}} = \mathcal{H} - \mu_\alpha J_\alpha^0, \quad (2.6)$$

where  $\mathcal{H}$  is the usual Hamiltonian density. Here and in the following, greek indices are always summed from 1 to  $N/2$ .

The canonical momenta which follow from (2.2) are

$$\Pi_a = \dot{\Phi}_a, \quad (2.7)$$

and we have

$$\mathcal{H} = \frac{1}{2}\Pi_a\Pi_a + \frac{1}{2}|\nabla\Phi|^2 + \frac{1}{2}(m^2 + \xi R)|\Phi|^2 + V_{tree}(\Phi) . \quad (2.8)$$

The generalized Hamiltonian density follows simply from (2.4)-(2.8). Here  $|\nabla\Phi|^2 = g^{ij}(\partial_i\Phi_a)(\partial_j\Phi_a)$ .

The reason for adopting a Hamiltonian approach is, that this is the easiest way to incorporate finite temperature effects in quantum field theory. First of all perform a Wick rotation  $\tau = it$  to obtain a  $(D+1)$ -dimensional Riemannian spacetime. Since we have restricted the spacetime to be ultrastatic, this presents no difficulty. The grand partition function is expressed as

$$\mathcal{Z} = \int [d\Pi_a][d\Phi_a] \left[ \int_0^\beta d\tau \int_\Sigma d\sigma_x \{ i\Pi_a \dot{\Phi}_a - \bar{\mathcal{H}} \} \right]. \quad (2.9)$$

The path integral in (2.9) extends over all fields  $\Phi_a$  periodic in time with period  $\beta = 1/T$ . Because  $\bar{\mathcal{H}}$  is quadratic in the momenta, the integration over the momenta in (2.9) may be performed simply by completing the square [17]. This leaves a configuration space path integral

$$\mathcal{Z} = \int [d\Phi_a] \exp(-\bar{S}[\Phi]), \quad (2.10)$$

where

$$\begin{aligned} \bar{S}[\Phi] = & \int_0^\beta d\tau \int_\Sigma d\sigma_x \left\{ \frac{1}{2} \left( \dot{\Phi}_{2\alpha-1} - i\mu_\alpha \Phi_{2\alpha} \right)^2 + \frac{1}{2} \left( \dot{\Phi}_{2\alpha} + i\mu_\alpha \Phi_{2\alpha-1} \right)^2 \right. \\ & \left. - \frac{1}{2} |\nabla\Phi|^2 - \frac{1}{2} (m^2 + \xi R) |\Phi|^2 - V_{tree}(\Phi) \right\}. \end{aligned} \quad (2.11)$$

Rather than deal with the thermodynamic potential, we will consider the finite temperature effective action computed using the background field method [18]. Thus we introduce some scalar background field  $\bar{\Phi}$ . By expanding about this background in the usual way, the one-loop effective action turns out to be

$$\begin{aligned}\Gamma &= \bar{S}[\bar{\Phi}] + \frac{1}{2} \ln \det \left\{ \lambda^2 \bar{S}_{ij}[\bar{\Phi}] \right\} \\ &= \bar{S}[\bar{\Phi}] + \Gamma^{(1)},\end{aligned}\tag{2.12}$$

where

$$\bar{S}_{ij}[\bar{\Phi}] = \frac{\delta^2 \bar{S}[\bar{\Phi}]}{\delta \bar{\Phi}_i(x) \delta \bar{\Phi}_j(x')}.\tag{2.13}$$

The second term  $\Gamma^{(1)}$  in (2.12) arises from the Gaussian functional integration, and represents the one-loop quantum correction to the classical action functional.  $\lambda$  is a unit of length introduced to keep the argument of the logarithm dimensionless. Using (2.11), for  $\bar{S}_{ij}[\bar{\Phi}]$  we find

$$\bar{S}_{ij}[\bar{\Phi}] = (M_{i,j} + V_{ij})\delta(x, x'),\tag{2.14}$$

where

$$\begin{pmatrix} M_{2\alpha-1, 2\alpha-1} & M_{2\alpha-1, 2\alpha} \\ M_{2\alpha, 2\alpha-1} & M_{2\alpha, 2\alpha} \end{pmatrix} = \begin{pmatrix} -\square - \mu_\alpha^2 + m^2 + \xi R & 2i\mu_\alpha \frac{\partial}{\partial \tau} \\ -2i\mu_\alpha \frac{\partial}{\partial \tau} & -\square - \mu_\alpha^2 + m^2 + \xi R \end{pmatrix},\tag{2.15}$$

the other components are zero, and

$$V_{ij} = V''_{tree}(\bar{\Phi})e_i e_j + \frac{V'_{tree}(\bar{\Phi})}{\bar{\Phi}}(\delta_{ij} - e_i e_j).\tag{2.16}$$

Here we introduced  $e_i = \bar{\Phi}_i/|\bar{\Phi}|^2$ . The separation in  $M$  and  $V$  is a separation in free field plus terms arising due to interaction.

In order to calculate the functional determinant of the operator  $\bar{S}_{ij}[\bar{\Phi}]$ , we will use the zeta function definition introduced by Hawking [19] and by Critchley, Dowker [20]. Then (2.12) reads

$$\Gamma = \bar{S}[\bar{\Phi}] - \frac{1}{2} \left[ \zeta'(0) + \zeta(0) \ln \lambda^2 \right],\tag{2.17}$$

where  $\zeta(s)$  is the zeta function associated with the operator  $\bar{S}_{ij}[\bar{\Phi}]$ , equation (2.14).

We are especially interested in the high temperature behaviour of the theory. In that regime heat-kernel techniques have been shown to be a very powerful tool [21, 14] and will also be used here. In doing that, one of the encountered problems is the diagonalization of  $\bar{S}_{ij}[\bar{\Phi}]$  [15, 22]. For  $\mu_\alpha = 0$  this may be done [22] and we will first consider this case in detail.

### 3 $O(N)$ -model with vanishing chemical potentials

In considering vanishing chemical potentials  $\mu_\alpha = 0$ , the problem is enormously simplified, because the operator  $M$ , (2.15), is diagonal,  $M_{ij} = -\delta_{ij}(\square + \xi R + m^2)$ . For that case, the eigenvalues of the operator  $\bar{S}_{ij}[\bar{\Phi}]$  may be represented in the form

$$\nu_{n,j} = \left(\frac{2\pi}{\beta}\right)^2 n^2 + \lambda_j, \quad (3.1)$$

with the eigenvalues  $\lambda_j$  of the operator

$$D = -\Delta + U(\bar{\Phi}). \quad (3.2)$$

For simplicity we introduced here

$$U_{ij}(\bar{\Phi}) = V_{ij} + (\xi R + m^2)\delta_{ij}.$$

Using a Mellin transformation, the zeta function of  $\bar{S}_{ij}[\bar{\Phi}]$  may be written as

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \sum_j \int_0^\infty dt t^{s-1} \exp \left\{ - \left[ \left(\frac{2\pi}{\beta}\right)^2 n^2 + \lambda_j \right] t \right\} \\ &= \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \int_0^\infty dt t^{s-1} \exp \left\{ - \left(\frac{2\pi}{\beta}\right)^2 n^2 t \right\} \text{Tr } K(x, x, t), \end{aligned} \quad (3.3)$$

where the kernel  $K(x, x', t)$  satisfies the equation

$$\begin{aligned} -\frac{\partial}{\partial t} K(x, x', t) &= (-\Delta + U(\bar{\Phi})) K(x, x', t), \\ \lim_{t \rightarrow 0} K(x, x', t) &= |g|^{-\frac{1}{2}} \delta(x, x'). \end{aligned} \quad (3.4)$$

We would like to concentrate on the high-temperature behaviour of the theory. Then for  $n \neq 0$  only small values of the parameter  $t$  contribute considerably to the integral in (3.3). So it is reasonable for  $n \neq 0$  to make use of the asymptotic behaviour for  $t \rightarrow 0$  of  $K(x, x', t)$  [23]. As mentioned, in this section we will not assume that the background field  $\bar{\Phi}$  is a constant. However, in order that the expansion that we are going to derive is reasonable,  $\bar{\Phi}$  has to be slowly varying, together with the condition  $\beta|R|^{\frac{1}{2}} \ll 1$ , where  $|R|$  is the magnitude of a typical curvature of the spacetime.

The suitable tool to derive the described expansion is the use of the following ansatz for the heat-kernel suggested by Parker and Toms [24],

$$\begin{aligned} K(x, x', t) &= \left(\frac{1}{4\pi t}\right)^{\frac{D}{2}} \exp \left\{ -\frac{\sigma(x, x')}{2t} - \left[ U(\bar{\Phi}) - \frac{1}{6}R \right] t \right\} \times \\ &\quad \Delta_{VM}(x, x') \Omega(x, x', t), \end{aligned} \quad (3.5)$$

where  $2\sigma(x, x')$  is the square of the proper arc length along the geodesic from  $x'$  to  $x$  and  $\Delta_{VM}(x, x')$  is the Van Vleck-Morette determinant. For  $t \rightarrow 0$  the function  $\Omega(x, x', t)$  may be expanded in an asymptotic series,

$$\Omega(x, x', t) = \sum_{l=0}^{\infty} a_l(x, x') t^l, \quad (3.6)$$

where the coefficients have to fulfill some recurrence relation. Using the given ansatz, (3.5), it has been shown by Jack and Parker [25], that the dependence of  $a_l$ ,  $l = 1, \dots, \infty$ , on the field  $\bar{\Phi}$  is only through derivatives of the field. As a result, combining (3.3), (3.5) and (3.6), we will automatically find a derivative expansion in the field  $\bar{\Phi}$  without any effort.

Although the method in principle works to every order, it gets very cumbersome after the leading terms. For that reason we concentrate from the beginning only on the leading terms. However, the method will be clear after that.

First of all one has

$$\zeta(s) = \zeta_{\Sigma}(s) + \zeta_z(s), \quad (3.7)$$

with the zeta function  $\zeta_{\Sigma}$  of the spatial section,

$$\zeta_{\Sigma}(x) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \text{Tr } K(x, x, t), \quad (3.8)$$

resulting from the  $n = 0$  term, and the finite temperature part

$$\zeta_z(s) = \frac{1}{\Gamma(s)} \sum'_{n=-\infty}^{\infty} \int_0^{\infty} dt t^{s-1} \exp \left\{ - \left( \frac{2\pi}{\beta} \right)^2 n^2 t \right\} \text{Tr } K(x, x, t), \quad (3.9)$$

where the prime indicates that the  $n = 0$  term of the sum is omitted. The zero temperature contribution represented by (3.8) has been analyzed in detail in [22] using an adiabatic expansion method, so let us concentrate on the finite temperature part, eq. (3.9). Use of the heat-kernel expansion (3.5) leads to

$$\zeta_z(s) = \sum_{i=0}^{\infty} \zeta_z^i(s) \quad (3.10)$$

with

$$\begin{aligned} \zeta_z^i(s) = & \frac{1}{(4\pi)^{\frac{D}{2}} \Gamma(s)} \sum'_{n=-\infty}^{\infty} \int_0^{\infty} dt t^{s-1-\frac{D}{2}} \exp \left\{ - \left( \frac{2\pi}{\beta} \right)^2 n^2 t \right\} \\ & \text{Tr} \left\{ \exp \left[ - (U(\bar{\Phi}) - \frac{1}{6} R) t \right] a_i t^i \right\}. \end{aligned} \quad (3.11)$$

We will concentrate only on the first two terms,  $\zeta_z^0(s)$  and  $\zeta_z^2(s)$  ( $\zeta_z^1(s)$  is zero due to  $a_1 = 0$ ), containing the leading terms of the high-temperature expansion. As one realizes,

a diagonalization of the potential matrix  $U(\bar{\Phi})$  is needed. For completeness, the matrix  $S$  accomplishing the diagonalization is presented in the Appendix A (for details see [22]). Using the results presented there, one first finds

$$\zeta_z^0(s) = \frac{2}{(4\pi)^{\frac{D}{2}} \Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} dt t^{s-1-\frac{D}{2}} \exp \left\{ - \left( \frac{2\pi}{\beta} \right)^2 n^2 t \right\} \times \\ \left[ \exp\{-M_1^2 t\} + (N-1) \exp\{-M_2^2 t\} \right], \quad (3.12)$$

where we introduced the effective masses

$$M_1^2 = m^2 + \left( \xi - \frac{1}{6} \right) R + V_{tree}''(\bar{\Phi}), \quad (3.13)$$

$$M_2^2 = m^2 + \left( \xi - \frac{1}{6} \right) R + \frac{V_{tree}'(\bar{\Phi})}{\bar{\Phi}}. \quad (3.14)$$

These masses, or more detailed  $(M_i \beta)^2$ , play the role of a natural expansion parameter of the theory. Performing the integration in (3.12) leads to Epstein type zeta functions,

$$E_1^{c^2} = \sum_{l=1}^{\infty} [l^2 + c^2]^{-\nu} \\ = \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(s+l)}{l! \Gamma(s)} \zeta_R(2s+2l) c^{2l}, \quad (3.15)$$

where the last equality holds for  $|c^2| < 1$  [26].  $\zeta_R(s)$  is the Riemann zeta function. With this, in principle all terms in the high temperature expansion may be calculated.

However, for simplicity let us present only the case  $D = 3$  and only the three leading terms. The relevant results then read

$$\zeta_z^0(0) = \frac{\beta}{2(4\pi)^2} \int_{\Sigma} d\sigma_x [M_1^4 + (N-1)M_2^4] \quad (3.16)$$

and

$$\zeta_z'^0(0) = \frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\Sigma} d\sigma_x \left\{ \frac{\sqrt{\pi}}{45} N \left( \frac{2\pi}{\beta} \right)^3 - \frac{\sqrt{\pi}}{3} \left( \frac{2\pi}{\beta} \right) [M_1^2 + (N-1)M_2^2] \right. \\ \left. + \frac{\beta}{2\sqrt{\pi}} \left[ \gamma + \ln \left( \frac{\beta}{4\pi} \right) \right] [M_1^4 + (N-1)M_2^4] \right. \\ \left. \mathcal{O}((M_i \beta)^6) \right\}. \quad (3.17)$$

For  $\zeta_z^2(s)$  we continue as indicated in the Appendix A, eq. (A.3). Including terms only up to  $\beta^{-3} \mathcal{O}((M_i \beta)^4)$ , the result reads

$$\zeta_z^2(s) = \frac{2}{(4\pi)^{\frac{3}{2}}} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)} \left( \frac{\beta}{2\pi} \right)^{2s+1} \zeta_R(2s+1) \int_{\Sigma} d\sigma_x A \quad (3.18)$$



with

$$A = \frac{N}{180}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}) - \frac{N}{6}\left(\xi - \frac{1}{5}\right)\square R - \frac{N-1}{6}\square\left(\frac{V'_{tree}(\bar{\Phi})}{\bar{\Phi}}\right) - \frac{1}{6}\square V''_{tree}(\bar{\Phi}). \quad (3.19)$$

This gives the contributions

$$\zeta_z^2(0) = \frac{1}{2(4\pi)^{\frac{3}{2}}}\frac{\beta}{\sqrt{\pi}}\int_{\Sigma}d\sigma_x A \quad (3.20)$$

and

$$\zeta'^2_z(0) = \frac{1}{(4\pi)^{\frac{3}{2}}}\frac{\beta}{\sqrt{\pi}}\left[\gamma + \ln\left(\frac{\beta}{4\pi}\right)\right]\int_{\Sigma}d\sigma_x A. \quad (3.21)$$

Dealing with a manifold without boundary, the terms in  $A$  containing the Laplace-Beltrami operator vanish.

Putting things together, we find for the high-temperature behaviour of the  $O(N)$ -model the result

$$\begin{aligned} \Gamma^{(1)} = & -\frac{1}{2(4\pi)^{\frac{3}{2}}}\int_{\Sigma}d\sigma_x\left\{\frac{\sqrt{\pi}}{45}N\left(\frac{2\pi}{\beta}\right)^3 - \frac{\sqrt{\pi}}{3}\frac{2\pi}{\beta}[M_1^2 + (N-1)M_2^2] \right. \\ & + \frac{\beta}{2\sqrt{\pi}}\left[\gamma + \ln\left(\frac{\beta\lambda}{4\pi}\right)\right][M_1^4 + (N-1)M_2^4 + 2A] \\ & \left. + \beta^{-3}\mathcal{O}((M_i\beta)^6)\right\} - \frac{1}{2}\zeta'_z(0). \end{aligned} \quad (3.22)$$

For the case  $N = 1$  it reduces to the results found in [27], as it should.

Let us now consider the case  $\mu_\alpha \neq 0$ . Because we will indicate how to develop a systematic approach in section 6 for the case of a general gauge group, here only some comments. Starting point of the calculation is

$$\zeta(s) = \frac{1}{\Gamma(s)}\sum_{n=-\infty}^{\infty}\int_0^{\infty}dt t^{s-1}\text{Tr}\left\{\exp[\bar{S}_{ij}[\bar{\Phi}]t]\right\} \quad (3.23)$$

with  $\bar{S}_{ij}[\bar{\Phi}]$  given in (2.14). As seen in (2.15) and (2.16),  $\bar{S}_{ij}[\bar{\Phi}]$  splits into the free case and, for small coupling, into small corrections due to the self-interaction. Treating the self-interaction using normal quantum-field perturbation theory, the leading term clearly gives the theory of  $N/2$  charged scalar fields with associated chemical potentials  $\mu_\alpha$  and the effective potential, or up to the order we write it down also the thermodynamic potential, for the  $O(N)$  model at finite  $T$  and finite  $\mu_\alpha$  reads,

$$\begin{aligned} \Omega = & -\frac{\pi^2}{90}NV T^4 - \frac{1}{6}T^2V\sum_{\alpha=1}^{N/2}\mu_\alpha^2 \\ & + \frac{1}{24}T^2\int_{\Sigma}d\sigma_x[M_1^2 + (N-1)M_2^2] + \dots \end{aligned} \quad (3.24)$$

In section 6 we will generalize this formula from the gauge group  $O(N)$  to an arbitrary rigid gauge group  $G$ .

## 4 $O(2)$ model at finite chemical potential

Let us now concentrate on the  $O(2)$  model with a constant classical background field. For  $N = 2$  the analysis is somehow more explicit and systematic approaches may be developed at finite background charge. We will present two of them. One is based on an expansion around  $\mu = 0$ , the second one is an expansion in the coupling constant which is assumed to be small, this is in powers of the potential  $V_{tree}$  and its derivatives, as indicated at the end of section III. In both cases dimensional analysis shows, that no other type of contributions may appear and that the expansion up to a certain order is complete. In this section we will also include the possibility that the manifold has a boundary.

### 4.1 Exansion around $\mu = 0$

One possibility to treat the  $O(2)$  model in curved spacetime is to follow the approach developed for the theory in the Minkowski spacetime [5]. In detail the fluctuation operator is given by

$$\begin{aligned}
\bar{S}_{11} &= \delta^4(x-y) \left[ -\square - \mu^2 + U''(\Phi) \frac{\Phi_1^2}{\Phi^2} + \frac{U'(\Phi)}{\Phi} \frac{\Phi_2^2}{\Phi^2} \right]_{\Phi=\bar{\Phi}}, \\
\bar{S}_{12} &= \delta^4(x-y) \left[ 2i\mu \frac{\partial}{\partial \tau} + U''(\Phi) \frac{\Phi_1 \Phi_2}{\Phi^2} - \frac{U'(\Phi)}{\Phi} \frac{\Phi_1 \Phi_2}{\Phi^2} \right]_{\Phi=\bar{\Phi}}, \\
\bar{S}_{21} &= \delta^4(x-y) \left[ -2i\mu \frac{\partial}{\partial \tau} + U''(\Phi) \frac{\Phi_1 \Phi_2}{\Phi^2} - \frac{U'(\Phi)}{\Phi} \frac{\Phi_1 \Phi_2}{\Phi^2} \right]_{\Phi=\bar{\Phi}}, \\
\bar{S}_{22} &= \delta^4(x-y) \left[ -\square - \mu^2 + U''(\Phi) \frac{\Phi_2^2}{\Phi^2} + \frac{U'(\Phi)}{\Phi} \frac{\Phi_1^2}{\Phi^2} \right]_{\Phi=\bar{\Phi}},
\end{aligned} \tag{4.1}$$

with  $U(\bar{\Phi}) = (1/2)(m^2 + \xi R)\bar{\Phi}^2 + V_{tree}(\bar{\Phi})$ .

Doing the Gaussian integration, the one-loop approximation of the partition sum is

$$\begin{aligned}
\ln \mathcal{Z}^{(1)} &= -\frac{1}{2} \text{tr} \ln \left\{ \square^2 - 4\mu^2 \partial_\tau^2 - \left( U''(\Phi) + \frac{U'(\Phi)}{\Phi} - 2\mu^2 \right) \square \right. \\
&\quad \left. + \left( U''(\Phi) - \mu^2 \right) \left( \frac{U'(\Phi)}{\Phi} - \mu^2 \right) \right\}_{\Phi=\bar{\Phi}}.
\end{aligned} \tag{4.2}$$

Introducing

$$\gamma^2 = \left( -\Delta + U''(\bar{\Phi}) - \mu^2 \right) \left( -\Delta + \frac{U'(\bar{\Phi})}{\bar{\Phi}} - \mu^2 \right) \tag{4.3}$$

and

$$\alpha^2 = 2 \left[ -\Delta + \mu^2 + \frac{1}{2} \left( U''(\bar{\Phi}) + \frac{U'(\bar{\Phi})}{\bar{\Phi}} \right) - \gamma \right], \quad (4.4)$$

this may be rewritten as

$$\ln \mathcal{Z}^{(1)} = \frac{1}{2} \text{tr} \ln \left\{ (-\partial_\tau^2 + \alpha \partial_\tau + \gamma)(-\partial_\tau^2 - \alpha \partial_\tau + \gamma) \right\}. \quad (4.5)$$

Assuming now a complete basis of eigenfunctions of the Laplace-Beltrami operator  $-\Delta$  with eigenvalues  $\lambda_n$  and denoting the resulting quantities of eqs. (4.3), (4.4), as  $\gamma_n^2$  and  $\alpha_n^2$ , this results formally in

$$\ln \mathcal{Z}^{(1)} = E_{\text{zero-point}} + \Omega, \quad (4.6)$$

where we introduced the infinite zero-point energy,

$$E_{\text{zero-point}} = \sum_j \frac{E_{+,j} + E_{-,j}}{2}, \quad (4.7)$$

and the thermodynamic potential

$$\Omega = \frac{1}{\beta} \sum_j \left[ \ln \left( 1 - e^{-\beta E_{+,j}} \right) + \ln \left( 1 - e^{-\beta E_{-,j}} \right) \right]. \quad (4.8)$$

Furthermore, the excitation energies are

$$E_{\pm,n} = \sqrt{\gamma_n + \frac{\alpha_n^2}{4}} \pm \frac{\alpha_n}{2}. \quad (4.9)$$

The representation eqs. (4.7), (4.8), of the one-loop grand partition sum is completely analogous to the case of a free scalar field in curved spacetime, the only change is in the excitation energies. Putting  $U(\Phi) = (1/2)(m^2 + \xi R)\Phi^2$ , this is the free field, the grand thermodynamic potential of a free field propagating in curved background is thus found directly.

Also for  $N = 2$ , in general it will be impossible or at least very difficult to treat the one-loop contribution, eq. (4.6), exactly. However, we are interested in the Bose-Einstein condensation and the relevant range for this phenomenon is the high-temperature regime to which we will once more concentrate.

By the mentioned analogy with the free field, one may obtain without any calculation the high-temperature expansion in terms of quantities given through the zeta-function of the excitation energies,

$$\zeta_{\pm}(\nu) = \sum_j (E_{\pm,j}^2)^{-\nu}, \quad (4.10)$$

or the associated heat-kernel coefficients,

$$K_{\pm}(t) = \sum_j e^{-E_{\pm,j}^2 t} \sim \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \sum_{l=0,1/2,\dots}^{\infty} b_l^{\pm} t^l. \quad (4.11)$$

By introducing  $E_{\pm,zeropoint}$ ,  $\Omega_{\pm}$ , as the quantities in (4.7) and (4.8) resulting from the excitation energies  $E_{\pm,j}$ , the results read

$$\begin{aligned} \Omega_{\pm} = & -\frac{1}{2}PP\zeta_{\pm}(-\frac{1}{2}) + \frac{1}{(4\pi)^2} \\ & \times \left\{ -b_2^{\pm} \ln\left(\frac{\beta}{2\pi}\right) + \psi(2) + \frac{2\sqrt{\pi}}{\beta} b_{\frac{3}{2}}^{\pm} \ln\beta + P_{\pm} + S_{\pm} \right\}, \end{aligned} \quad (4.12)$$

with

$$S_{\pm} = - \sum_{r=1/2,1,\dots}^{\infty} b_{2+r}^{\pm} \left(\frac{\beta}{4\pi}\right)^{2r} \frac{(2r)!}{\Gamma(r+1)} \zeta_R(1+2r) \quad (4.13)$$

and

$$P_{\pm} = - \sum_{r=0,1/2,1} b_r^{\pm} \left(\frac{\beta}{2}\right)^{-4+2r} \Gamma(2-r) \zeta_R(4-2r). \quad (4.14)$$

Here,  $PP\zeta_{\pm}(-1/2)$  denotes the finite part of  $\zeta_{\pm}(s)$  at  $s = -1/2$ .

Thus we have reduced the task of determining the high-temperature behaviour of the theory to an analytical treatment of  $\zeta_{\pm}(\nu)$ , eq. (4.10).

However, due to the very complicated structure of the excitation energies, eq. (4.9), this is also a very difficult task. Even for the determination of the heat-kernel coefficients, eq. (4.11), it is not clear how to do it for the coefficients  $b_{\pm}$  separately. To explain the difficulties, let us write  $\lambda_j$  for the eigenvalues of the Laplacian, then we have

$$\begin{aligned} E_{\pm,j}^2 = & \lambda_j + \mu^2 + \frac{1}{2} \left[ U''(\Phi) + \frac{U'(\Phi)}{\Phi} \right] \Big|_{\Phi=\bar{\Phi}} \\ & \pm \left[ 2\mu^2 \left\{ 2\lambda_j + \left[ U''(\Phi) + \frac{U'(\Phi)}{\Phi} \right] \Big|_{\Phi=\bar{\Phi}} \right\} + \frac{1}{4} \left[ U''(\Phi) - \frac{U'(\Phi)}{\Phi} \right]^2 \Big|_{\Phi=\bar{\Phi}} \right]^{\frac{1}{2}}. \end{aligned} \quad (4.15)$$

Here the difference to the case of vanishing background charge is seen explicitly. In contrast to a second order elliptic differential operator (to which eq. (4.15) reduces for  $\mu = 0$ ) in this case one has a pseudo-differential operator. Whereas in the first case one could use very powerful results of mathematicians and physisists for this kind of operators, to our knowledge no results on the heat-kernel expansion, eq. (4.11), are available. However, as we will see in the actually needed sum of  $\Omega_+$  and  $\Omega_-$  we avoid the mentioned problem and the calculation of the coefficient  $b_l^{\pm}$  reduces to the known case  $\mu = 0$ .

For notational convenience let us write

$$\begin{aligned} d_j &= \lambda_j + \frac{1}{2} \left[ U''(\Phi) + \frac{U'(\Phi)}{\Phi} \right] \Big|_{\Phi=\bar{\Phi}}, \\ b &= \frac{1}{4} \left[ U''(\Phi) - \frac{U'(\Phi)}{\Phi} \right]^2 \Big|_{\Phi=\bar{\Phi}}, \end{aligned}$$

thus the eigenvalues read

$$E_{\pm,j} = d_j + \mu^2 \pm \sqrt{4\mu^2 d_j + b}. \quad (4.16)$$

As we see in eq. (4.8), we need the asymptotic  $t \rightarrow 0$  expansion of

$$\begin{aligned} K(t) &= K_+(t) + K_-(t) \\ &\sim \left( \frac{1}{4\pi t} \right)^{\frac{3}{2}} \sum_{l=0, \frac{1}{2}, 1, \dots}^{\infty} (b_l^+ + b_l^-) t^l. \end{aligned} \quad (4.17)$$

Expansion of the exponential factor containing the square root leads to the cancellation of terms containing explicitly a square root, letting us with

$$K(t) \sim 2 \sum_j \sum_{l=0}^{\infty} e^{-(d_j + \mu^2)t} \frac{t^{2l}}{(2l)!} (4\mu^2 d_j + b)^l. \quad (4.18)$$

The powers in  $d_j$  may be produced by differentiation of the exponential, which then yields a reduction to the heat-kernel connected with  $d_j$  which is nowadays a very well known problem. In doing that, it is seen, that the leading term for  $t \rightarrow 0$  of the summation index  $l$  is given by  $t^{-\frac{3}{2}+l}$ . Thus, reducing our attention to the coefficients  $b_0, \dots, b_2, b_l = b_l^+ + b_l^-$ , we find that the following terms may contribute

$$K(t) = A_0 + A_1 + A_2 + \mathcal{O}(t) \quad (4.19)$$

with

$$\begin{aligned} A_0 &= 2 \left( 1 - \mu^2 t + \frac{1}{2} [b + \mu^4] t^2 + \mathcal{O}(t^3) \right) \sum_j e^{-d_j t}, \\ A_1 &= 4\mu^2 t^2 \left( 1 - \mu^2 t + \mathcal{O}(t^2) \right) \sum_j d_j e^{-d_j t}, \\ A_2 &= \frac{4}{3} \mu^4 t^4 (1 + \mathcal{O}(t)) \sum_j d_j^2 e^{-d_j t}. \end{aligned} \quad (4.20)$$

Introducing the kernel of  $d_j$ ,

$$B(t) = \sum_j e^{-a_j t} \sim \left( \frac{1}{4\pi t} \right)^{\frac{3}{2}} \sum_{l=0, \frac{1}{2}, 1, \dots} c_l t^l, \quad (4.21)$$

this may be rewritten as

$$\begin{aligned}
A_0 &= \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \times 2 \left( c_0 + c_{\frac{1}{2}} t^{\frac{1}{2}} + [c_1 - c_0 \mu^2] t + [c_{\frac{3}{2}} - \mu^2 c_{\frac{1}{2}}] t^{\frac{3}{2}} \right. \\
&\quad \left. + [c_2 - \mu^2 c_1 + \frac{1}{2}(b + \mu^4) c_0] t^2 + \mathcal{O}(t^{\frac{5}{2}}) \right), \\
A_1 &= \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \times \left( 6c_0 \mu^2 t + 4\mu^2 c_{\frac{1}{2}} t^{\frac{3}{2}} + 2[c_1 \mu^2 - 3c_0 \mu^4] t^2 + \mathcal{O}(t^{\frac{5}{2}}) \right), \\
A_2 &= \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \times \left( 5\mu^4 c_0 t^2 + \mathcal{O}(t^{\frac{5}{2}}) \right).
\end{aligned} \tag{4.22}$$

Using the knowledge of the heat-kernel coefficients  $c_j$ , this solves already the task of finding the high-temperature behaviour of the considered theory and it can be done in principle to any desired order. Summing up the contributions in equation (4.22) we obtain to the considered order

$$\begin{aligned}
K(t) \sim \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \Big\{ & 2c_0 + 2c_{\frac{1}{2}} t^{\frac{1}{2}} + 2[c_1 + 2c_0 \mu^2] t \\
& + 2[c_{\frac{3}{2}} + c_{\frac{1}{2}} \mu^2] t^{\frac{3}{2}} + 2\left[c_2 + \frac{1}{2} b c_0\right] t^2 + \mathcal{O}(t^{\frac{5}{2}}) \Big\}.
\end{aligned} \tag{4.23}$$

Thus the leading terms of the thermodynamic potential read

$$\begin{aligned}
\Omega \sim & -\frac{\pi^2}{45} c_0 T^4 - \frac{1}{2\pi^{\frac{3}{2}}} c_{\frac{1}{2}} \zeta_R(3) T^3 \\
& - \frac{1}{12} [c_1 + 2c_0 \mu^2] T^2 + \mathcal{O}(T)
\end{aligned} \tag{4.24}$$

For a manifold without boundary this is for the effective potential,

$$V_{thermal} \sim -\frac{\pi^2}{45} T^4 - \frac{1}{6} \mu^2 T^2 + \frac{1}{12} M^2 T^2 + \mathcal{O}(T) \tag{4.25}$$

with

$$M^2 = \frac{1}{2} \left[ U''(\Phi) + \frac{U'(\Phi)}{\Phi} \right] \Big|_{\Phi=\bar{\Phi}} - \frac{1}{6} R.$$

For the interaction potential

$$U(\Phi) = \frac{1}{2} (m^2 + \xi R) \Phi^2 + \frac{\lambda}{4!} \Phi^4$$

we have

$$M^2 = m^2 + \left( \xi - \frac{1}{6} \right) R + \frac{1}{3} \lambda \bar{\Phi}^2$$

generalizing the flat space result [2, 5, 4] to curved spacetimes.

## 4.2 Weak coupling expansion

Our second approach is based on an expansion in the self-interaction potential. In doing that, let us also present another very simple way to determine the excitation energies  $E_{\pm}$ . For convenience we introduce here a complex field  $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$ . Then, the action (2.11) is given by

$$\begin{aligned} \bar{S} = & \int dt \int_{\Sigma} d\sigma_x \left\{ (\dot{\Phi}^\dagger + i\mu\Phi^\dagger)(\dot{\Phi} - i\mu\Phi) - |\nabla\Phi|^2 \right. \\ & \left. - (m^2 + \xi R)|\Phi|^2 - V_{tree}(\Phi) \right\}. \end{aligned} \quad (4.26)$$

The field equations for  $\Phi$  and  $\Phi^\dagger$  follow from  $\bar{S}$  in the usual way by varying with respect to  $\Phi$  and  $\Phi^\dagger$  independently. It is easily seen that

$$0 = -\ddot{\Phi} + 2i\mu\dot{\Phi} + \mu^2\Phi + \nabla^2\Phi - (m^2 + \xi R)\Phi - V'_{tree}\Phi, \quad (4.27)$$

$$0 = -\ddot{\Phi}^\dagger + 2i\mu\dot{\Phi}^\dagger + \mu^2\Phi^\dagger + \nabla^2\Phi^\dagger - (m^2 + \xi R)\Phi^\dagger - V'_{tree}\Phi^\dagger. \quad (4.28)$$

(Here  $V'_{tree} = \frac{\partial}{\partial|\Phi|^2}V_{tree}(|\Phi|^2)$ .) The excitation energies are obtained by looking at small fluctuations about  $\bar{\Phi}$ . We will write

$$\Phi = \bar{\Phi} + \Psi \quad (4.29)$$

and linearize (4.27,4.28) in  $\Psi$ . This gives

$$\begin{aligned} 0 = & -\ddot{\Psi} + 2i\mu\dot{\Psi} + \mu^2\Psi + \nabla^2\Psi - (m^2 + \xi R)\Psi \\ & - V'_{tree}(|\bar{\Phi}|^2)\Psi - V''_{tree}(|\bar{\Phi}|^2)[|\bar{\Phi}|^2\Psi + \bar{\Phi}^2\Psi^\dagger] \end{aligned} \quad (4.30)$$

along with the equation obtained by taking the complex conjugate of (4.30). It is of course not possible to solve (4.30); however, we can expand  $\Psi$  and  $\Psi^\dagger$  in terms of a complete set of functions. Let  $\{f_N(\mathbf{x})\}$  be a complete set of eigenfunctions of  $-\nabla^2 + \xi R$  with eigenvalues  $\sigma_N$ :

$$(-\nabla^2 + \xi R)f_N(\mathbf{x}) = \sigma_N f_N(\mathbf{x}). \quad (4.31)$$

We can write

$$\Psi(t, \mathbf{x}) = \sum_N A_N e^{-iE_N t} f_N(\mathbf{x}), \quad (4.32)$$

$$\Psi^\dagger(t, \mathbf{x}) = \sum_N B_N e^{-iE_N t} f_N(\mathbf{x}), \quad (4.33)$$

for some independent expansion coefficients  $A_N$  and  $B_N$ . Substitution of (4.32,4.33) into (4.30) and its complex conjugate leads to the coupled equations

$$\begin{aligned} 0 = & \left[ (E_N + \mu)^2 - \sigma_N - m^2 - V'_{tree} - V''_{tree}|\bar{\Phi}|^2 \right] A_N \\ & - \bar{\Phi}^2 V''_{tree} B_N, \end{aligned} \quad (4.34)$$

$$\begin{aligned} 0 = & \left[ (E_N - \mu)^2 - \sigma_N - m^2 - V'_{tree} - V''_{tree}|\bar{\Phi}|^2 \right] B_N \\ & - (\bar{\Phi}^\dagger)^2 V''_{tree} A_N. \end{aligned} \quad (4.35)$$

In order for a non-trivial solution for  $A_N$  and  $B_N$  we must have  $E_N = E_{\pm N}$  where

$$E_{\pm N} = \left\{ \mu^2 + M_N^2 \pm \left[ 4\mu^2 M_N^2 + |\bar{\Phi}|^4 (V_{tree}'')^2 \right]^{1/2} \right\}^{1/2}, \quad (4.36)$$

with

$$M_N^2 = \sigma_N + m^2 + V_{tree}' + |\bar{\Phi}|^2 V_{tree}'' . \quad (4.37)$$

This gives the excitation energies of Refs. [5, 4] if the result is rewritten in terms of real fields and it is the result (4.9)

Once the excitation energies have been determined, the thermodynamic potential follows from the usual definition. Alternatively we can deal with the effective potential or effective action. The thermodynamic potential  $\Omega$  is given by

$$\Omega = T \sum_N \ln \left[ (1 - e^{-\beta E_{+N}}) (1 - e^{-\beta E_{-N}}) \right] . \quad (4.38)$$

(If we deal with the effective potential then there is a contribution from the zero-point energy [5], but it can be ignored at high temperature.) If we set  $V_{tree} = 0$  so that the theory is free, then it is easy to see from (4.36) and (4.37) that

$$E_{\pm N} = (\sigma_N + m^2)^{1/2} \pm \mu . \quad (4.39)$$

The term  $(\sigma_N + m^2)^{1/2}$  just gives the energy of a single mode  $f_N(\mathbf{x})$ , and is the analogue of the single particle energy  $(\mathbf{k}^2 + m^2)^{1/2}$  in flat spacetime. Using (4.39) in (4.38) results in the standard expression for the thermodynamic potential for a free boson gas at finite density.

When  $V_{tree} \neq 0$ , the evaluation of (4.38) becomes more complicated. A general approach to this problem can be made using generalized  $\zeta$ -functions, which has been presented in the first part of this section. However, it is very simple to obtain the leading term in  $\Omega$  coming from  $V_{tree}$  at high temperature. To do this define

$$M_N^2 = \omega_N^2 + \delta M^2 \quad (4.40)$$

where

$$\omega_N = (\sigma_N + m^2)^{1/2}, \quad (4.41)$$

$$\delta M^2 = V_{tree}'(|\bar{\Phi}|^2) + |\bar{\Phi}|^2 V_{tree}''(|\bar{\Phi}|^2) . \quad (4.42)$$

If we work to first order in the potential it is easy to show that

$$E_{\pm N} = \omega_N \pm \mu + \frac{1}{2\omega_N} \delta M^2 + \dots . \quad (4.43)$$

Using this in (4.38) and expanding to first order in  $\delta M^2$  results in

$$\begin{aligned} \Omega = & T \sum_N \ln \left( (1 - e^{-\beta(\omega_N + \mu)}) (1 - e^{-\beta(\omega_N - \mu)}) \right) \\ & + \sum_N \frac{1}{2\omega_N} \delta M^2 \left\{ \left[ e^{\beta(\omega_N + \mu)} - 1 \right]^{-1} \right. \\ & \left. + \left[ e^{\beta(\omega_N - \mu)} - 1 \right]^{-1} \right\} + \dots . \end{aligned} \quad (4.44)$$



The first term on the right hand side of (4.44) is just the thermodynamic potential in the absence of interactions,  $\Omega_{free}$ ,

$$\Omega_{free} = T \sum_N \ln \left( 1 - e^{-\beta(\omega_N + \mu)} \right) \left( 1 - e^{-\beta(\omega_N - \mu)} \right). \quad (4.45)$$

To evaluate the second term on the right hand side of (4.44), note that

$$\frac{\partial}{\partial m^2} \ln \left( 1 - e^{-\beta(\omega_N \pm \mu)} \right) = \frac{\beta}{2\omega_N} \left[ e^{\beta(\omega_N \pm \mu)} - 1 \right]^{-1}. \quad (4.46)$$

This shows that

$$\begin{aligned} \Omega_1 &= \sum_N \frac{1}{2\omega_N} \delta M^2 \left\{ \left[ e^{\beta(\omega_N + \mu)} - 1 \right]^{-1} + \left[ e^{\beta(\omega_N - \mu)} - 1 \right]^{-1} \right\} \\ &= \delta M^2 \frac{\partial}{\partial m^2} \Omega_{free}. \end{aligned} \quad (4.47)$$

The first order correction to the thermodynamic potential due to the interaction can be determined from a knowledge of  $\Omega_{free}$ .

Expressions for  $\Omega_{free}$  can be found in a number of places [21, 14, 13]. The result for a manifold with boundary is (for  $D = 3$ )

$$\begin{aligned} \Omega_{free} &= -\frac{\pi^2}{45} T^4 \theta_0 - \frac{1}{2} \pi^{-3/2} \zeta_R(3) T^3 \theta_{1/2} \\ &\quad - \frac{1}{12} T^2 \left[ \theta_1 - (m^2 - 2\mu^2) \theta_0 \right] + \dots \end{aligned} \quad (4.48)$$

at high temperature.  $\theta_k$  are the heat kernel coefficients for  $\text{tr exp} \left( -t(-\nabla^2 + \xi R) \right)$ . The result for spatial dimensions other than 3 is also easily obtained :

$$\begin{aligned} \Omega_{free} &= -2\pi^{-(D+1)/2} \Gamma((D+1)/2) \zeta_R(D+1) T^{D+1} \theta_0 \\ &\quad - \pi^{-(D+1)/2} \Gamma(D/2) \zeta_R(D) T^D \theta_{1/2} \\ &\quad - \frac{1}{2} \pi^{-(D+1)/2} \Gamma((D-1)/2) \zeta_R(D-1) T^{D-1} \\ &\quad \times \left\{ \theta_1 - \left[ m^2 - (D-1)\mu^2 \right] \theta_0 \right\} + \dots \end{aligned} \quad (4.49)$$

for  $D \geq 3$ .

If we consider only  $D = 3$ , and use  $\theta_0 = V$ , then

$$\frac{\partial}{\partial m^2} \Omega_{free} = \frac{1}{12} T^2 V + \dots \quad (4.50)$$

and so from (4.47)

$$\Omega_1 = \frac{1}{12} T^2 V \delta M^2 + \dots. \quad (4.51)$$

We therefore have found the high temperature expansion

$$\begin{aligned}\Omega &= -\frac{\pi^2}{45}VT^4 - \frac{1}{2}\pi^{-3/2}\zeta_R(3)T^3\theta_{1/2} \\ &\quad + \frac{1}{12}T^2[(\delta M^2 + m^2 - 2\mu^2)V - \theta_1] + \dots\end{aligned}\quad (4.52)$$

$\delta M^2$  is given directly in terms of the interaction potential by (4.42). If we specialize to  $\Sigma$  having no boundary then  $\theta_{1/2} = 0$  and  $\theta_1 = -(\xi - 1/6)RV$ . We therefore find

$$\begin{aligned}V_{thermal} &= \Omega/V \\ &= -\frac{\pi^2}{45}T^4 + \frac{1}{12}T^2[V'_{tree}(|\bar{\Phi}|^2) + |\bar{\Phi}|^2V''_{tree}(|\bar{\Phi}|^2) \\ &\quad + m^2 + (\xi - 1/6)R - 2\mu^2] + \dots\end{aligned}\quad (4.53)$$

in agreement with eq. (4.25). It is possible to calculate higher order terms in the interaction potential by obtaining the second order correction to the excitation energies. However it is clear on dimensional grounds that such terms will only involve lower powers of  $T$ , and therefore that these terms will be negligible at high temperature.

## 5 $O(N)$ model at finite chemical potential in the limit $N \rightarrow \infty$

Let us conclude the considerations on the  $O(N)$  model with two comments on the limit  $N \rightarrow \infty$ .

The first comment is on technical simplifications that occur. In the large- $N$  limit the off diagonal elements of the potential matrix  $V(\bar{\Phi})$ , eq. (2.16), may be neglected [2], so that the potential reduces to

$$V_{ij} = \delta_{ij} \left( \frac{V'_{tree}(\bar{\Phi})}{\bar{\Phi}} \right). \quad (5.1)$$

Thus the fluctuation operator  $\bar{S}_{ij}[\bar{\Phi}]$  splits into  $N/2$   $2 \times 2$  matrices,

$$\begin{aligned}\bar{S}_\alpha[\bar{\Phi}] &= \begin{pmatrix} \bar{S}_{2\alpha-1, 2\alpha-1} & \bar{S}_{2\alpha-1, 2\alpha} \\ \bar{S}_{2\alpha, 2\alpha-1} & \bar{S}_{2\alpha, 2\alpha} \end{pmatrix} \\ &= \begin{pmatrix} -\square - \mu_\alpha^2 + M_2^2 & 2i\mu_\alpha \frac{\partial}{\partial \tau} \\ -2i\mu_\alpha \frac{\partial}{\partial \tau} & -\square - \mu_\alpha^2 + M_2^2 \end{pmatrix}.\end{aligned}\quad (5.2)$$

So it is easily seen, that also the one-loop contribution splits into the sum of  $N/2$  contributions,

$$\ln \mathcal{Z}^{(1)} = -\frac{1}{2} \sum_{\alpha}^{N/2} \ln \det \bar{S}_\alpha[\bar{\Phi}]. \quad (5.3)$$

Thus in the limit of large  $N$ , the one-loop contribution is simply the sum of the contribution of  $N/2$  complex fields with their associated chemical potentials.

The second comment is on the influence of higher-loop contributions in the limit  $N \rightarrow \infty$ . As is well known, at the critical temperature the one-loop approximation is not valid any more. This is made obvious by the fact, that the exact one-loop contribution would lead to an unacceptable complex critical temperature [28]. The reason is that higher loop contribution become significant and all so called daisy diagrams have to be summed [28, 29]. Let us look in some detail at this summation for the  $O(N)$  theory in curved spacetime and let us restrict for definiteness on the  $\lambda\Phi^4$  self-interaction. Then in principle, the expansion around the background field leads also to odd powers in  $\Phi$ , but in the limit  $N \rightarrow \infty$  these are negligible [28]. The relevant quantity to tackle with the problem is the 1PI self-energy, which one may approximate by [29]

$$\sigma \sim \frac{\lambda TN}{6} \sum_{n=-\infty}^{\infty} \sum_l \frac{1}{(2\pi/\beta)^2 n^2 + \lambda_l}, \quad (5.4)$$

with the eigenvalues  $\lambda_l$  of the Laplace-Beltrami operator  $\Delta_\Sigma$ . Let us regularize the expression by considering

$$\zeta_\sigma(s) = \frac{\lambda TN}{6} \sum_{n=-\infty}^{\infty} \sum_l \left( \left( \frac{2\pi}{\beta} \right)^2 n^2 + \lambda_l \right)^{-s} \quad (5.5)$$

and let's derive a high-temperature expansion of that expression. Using the method described in section III, one arrives at

$$\begin{aligned} \zeta_\sigma(s) &= \frac{\lambda TN}{6} \zeta_\Delta(s) \\ &+ \frac{\lambda TN}{3(4\pi)^{\frac{3}{2}} \Gamma(s)} \sum_{l=0}^{\infty} \Gamma\left(s - \frac{3}{2} + l\right) a_l(\Delta) (2\pi T)^{3-2s-2l} \zeta_R(2s-3+2l), \end{aligned} \quad (5.6)$$

with the zeta function  $\zeta_\Delta$  and heat-kernel coefficients  $a_l(\Delta)$  of the operator  $\Delta_\Sigma$ . From general zeta function theory it follows, that depending on spacetime properties there might exist a pole at  $s = 1$ . This divergent piece has to be dealt with in the zero temperature renormalization theory. The leading terms of the finite part of the self-energy  $\sigma$  defined by the finite part  $PP$  of  $\zeta_\sigma(s)$ , are

$$\sigma \sim \frac{\lambda NT^2}{72} + \frac{\lambda NT}{6} PP \zeta_\Delta(1). \quad (5.7)$$

The leading term is exactly the flat spacetime result, as to be expected for dimensional reasons, and so also in curved spacetime the leading orders of the expansion are not influenced by summing over all daisy diagrams.

## 6 Extension to general gauge groups

We have, following Haber and Weldon [2], studied the  $O(N)$  model for  $N$  even. In this section we wish to consider the more general case of an arbitrary gauge group  $G$  with a set of interacting complex scalar fields with a rigid gauge invariance under the action of  $G$ .

Let  $\Phi$  denote the set of complex scalar fields transforming under some  $d_s$ -dimensional representation of  $G$ . We can write

$$\Phi \rightarrow U\Phi \quad (6.1)$$

where

$$U = \exp(i\theta^i T_i) \quad (6.2)$$

with  $T_i$  the hermitian generators for the Lie algebra of  $G$  taken in the  $d_s$ -dimensional representation for the scalar fields. We will use  $i = 1, 2, \dots, d_G$  where  $d_G$  is the dimension of  $G$ , and  $\alpha = 1, 2, \dots, d_s$ . Because we choose to work with complex fields rather than real fields here, the action is

$$S = \int dt \int_{\Sigma} d\sigma_x \left\{ (\partial^\mu \Phi^\dagger)(\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi - \xi R \Phi^\dagger \Phi - V_{tree} \right\} \quad (6.3)$$

where  $\Phi$  may be regarded as a column vector of complex scalar fields with  $d_s$  components.  $V_{tree}$  is assumed to be invariant under (6.1), and for simplicity we will assume that it only depends on  $\Phi$  and  $\Phi^\dagger$  through the combination  $|\Phi|^2 = \Phi^\dagger \Phi$  as before.

A set of conserved currents can be found in the usual way (see Ref. [30] for example) by looking at the change in (6.3) under an infinitesimal gauge transformation. It is easy to show that

$$J_j^\mu = i \left( \Phi^\dagger T_j \partial^\mu \Phi - \partial^\mu \Phi^\dagger T_j \Phi \right) \quad (6.4)$$

gives the conserved currents, and hence

$$Q_j = i \int_{\Sigma} d\sigma_x \left( \Phi^\dagger T_j \dot{\Phi} - \dot{\Phi}^\dagger T_j \Phi \right) \quad (6.5)$$

gives a set of conserved charges. For the case of  $O(N)$  these results reduce to those of Haber and Weldon [2] given in Sec. 2.

A chemical potential  $\mu^i$  can be introduced for each charge  $Q_i$ , and a Hamiltonian  $\bar{H}$  found which incorporates the conserved charges :

$$\bar{H} = H - \mu^i Q_i . \quad (6.6)$$

It is now possible to integrate out the field momenta in the Hamiltonian path integral as before to leave a path integral in configuration space. This results in

$$\begin{aligned} \bar{S} = \int dt \int_{\Sigma} d\sigma_x \left\{ \left( \dot{\Phi}^\dagger + i\mu^j \Phi^\dagger T_j \right) \left( \dot{\Phi} - i\mu^j T_j \Phi \right) - |\nabla \Phi|^2 \right. \\ \left. - (m^2 + \xi R) |\Phi|^2 - V_{tree} \right\} , \end{aligned} \quad (6.7)$$

as the replacement for (4.26). (If we specialize to  $O(N)$ , decompose the fields into real and imaginary parts, and perform a Wick rotation to imaginary time, then the earlier result in (2.11) is found.)

In order to calculate the thermodynamic potential we will content ourselves with a high temperature expansion. This can be done in two stages. We will first look at the case where the chemical potentials all vanish as we did in Sec. 3 for the  $O(N)$  model. However we will use the technique based on the excitation energies used in Sec. 4.2 for the  $U(1)$  model. After showing how the thermodynamic potential can be obtained in this case, we will then return to the case of non-vanishing chemical potentials and show how a perturbative expansion may be used to obtain the leading terms of the thermodynamic potential in the high temperature limit.

## 6.1 Vanishing chemical potentials

Setting  $\mu^i = 0$  in (6.7) and varying with respect to  $\Phi$  and  $\Phi^\dagger$  results in the field equation

$$0 = -\ddot{\Phi} + \nabla^2 \Phi - (m^2 + \xi R)\Phi - V'_{tree} \Phi , \quad (6.8)$$

and its complex conjugate. As in Sec. 4.2, let  $\Phi = \bar{\Phi} + \Psi$  and  $\Phi^\dagger = \bar{\Phi}^\dagger + \Psi^\dagger$  where  $\bar{\Phi}$  denotes the background field. Linearizing in  $\Psi$  results in

$$\begin{aligned} 0 = & -\ddot{\Psi} + \nabla^2 \Psi - (m^2 + \xi R)\Psi - V'_{tree}(|\bar{\Phi}|^2)\Psi \\ & - V''_{tree}(|\bar{\Phi}|^2)(\bar{\Phi}^\dagger \Psi + \Psi^\dagger \bar{\Phi})\bar{\Phi} , \end{aligned} \quad (6.9)$$

and its complex conjugate. By using expansions as in (4.32) and (4.33), except that  $A_N$  is a column vector and  $B_N$  is a row vector, we obtain the coupled set of equations

$$\begin{aligned} 0 = & [(E_N^2 - m^2 - \sigma_N - V'_{tree})\delta_{\alpha\beta} - V''_{tree}\bar{\Phi}_\beta^\dagger \bar{\Phi}_\alpha]A_{N\beta} \\ & - V''_{tree}\bar{\Phi}_\alpha \bar{\Phi}_\beta B_{N\beta} , \end{aligned} \quad (6.10)$$

$$\begin{aligned} 0 = & -V''_{tree}\bar{\Phi}_\alpha^\dagger \bar{\Phi}_\beta^\dagger A_{N\beta} + B_{N\beta}[(E_N^2 - m^2 - \sigma_N - V'_{tree})\delta_{\beta\alpha} \\ & - V''_{tree}\bar{\Phi}_\beta \bar{\Phi}_\alpha^\dagger] . \end{aligned} \quad (6.11)$$

One simple way of obtaining the energy spectrum is by decomposing

$$A_{N\beta} = A_{N\beta}^\perp + \bar{\Phi}_\beta A_N^\parallel \quad (6.12)$$

where

$$\bar{\Phi}_\beta^\dagger A_{N\beta}^\perp = 0 \quad (6.13)$$

and similarly

$$B_{N\beta} = B_{N\beta}^\perp + \bar{\Phi}_\beta^\dagger B_N^\parallel \quad (6.14)$$

where

$$\bar{\Phi}_\beta B_{N\beta}^\perp = 0 . \quad (6.15)$$

Using these decompositions in (6.10) and (6.11) results in

$$0 = aA_{N\alpha}^\perp + \bar{\Phi}_\alpha \{ (a - |\bar{\Phi}|^2 V''_{tree}) A_N^\parallel - |\bar{\Phi}|^2 V''_{tree} B_N^\parallel \} \quad (6.16)$$

$$0 = aB_{N\alpha}^\perp + \bar{\Phi}_\alpha^\dagger \{ (a - |\bar{\Phi}|^2 V''_{tree}) B_N^\parallel - |\bar{\Phi}|^2 V''_{tree} A_N^\parallel \} \quad (6.17)$$

where

$$a = E_N^2 - m^2 - \sigma_N - V'_{tree} . \quad (6.18)$$

By contracting (6.16) with  $\bar{\Phi}_\alpha^\dagger$  and (6.17) with  $\bar{\Phi}_\alpha$  it is easy to show that  $a = 0$  or  $a = 2|\bar{\Phi}|^2 V''_{tree}$ , where  $a = 0$  occurs with multiplicity  $d_s - 1$  and  $a = 2|\bar{\Phi}|^2 V''_{tree}$  occurs with multiplicity 1. The excitation energies are therefore

$$E_{+N} = [m^2 + \sigma_N + V'_{tree} + 2|\bar{\Phi}|^2 V''_{tree}]^{1/2} , \quad (6.19)$$

$$E_{-N} = [m^2 + \sigma_N + V'_{tree}]^{1/2} . \quad (6.20)$$

These results agree with the  $\mu \rightarrow 0$  limit of (4.36) and (4.37).

We therefore find that the thermodynamic potential for  $\mu = 0$  is

$$\Omega(\mu = 0) = T \sum_N \left\{ \ln(1 - e^{-\beta E_{+N}}) + (d_s - 1) \ln(1 - e^{-\beta E_{-N}}) \right\} . \quad (6.21)$$

It is now possible to make contact with the  $\zeta$ -function method of Secs. 2,3 to obtain the high temperature expansion. Instead of pursuing this we will simply obtain the leading order terms in the high temperature expansion using the weak-coupling limit described in Sec. 4.2. It is straightforward to show that

$$\begin{aligned} \Omega(\mu = 0) &\simeq T \sum_N \left\{ d_s \ln(1 - e^{-\beta \omega_N}) + \frac{\beta}{2\omega_N} (e^{\beta \omega_N} - 1)^{-1} \right. \\ &\quad \left. \times [d_s V'_{tree} + 2|\bar{\Phi}|^2 V''_{tree}] \right\} \end{aligned} \quad (6.22)$$

$$= \frac{1}{2} \{ d_s + [d_s V'_{tree} + 2|\bar{\Phi}|^2 V''_{tree}] \frac{\partial}{\partial m^2} \} \Omega_{free}(\mu = 0) \quad (6.23)$$

where

$$\Omega_{free}(\mu = 0) = T \sum_N \ln(1 - e^{-\beta \omega_N})^2 \quad (6.24)$$

is the thermodynamic potential for a single complex scalar field with no interactions in the absence of a chemical potential. In the case of  $D = 3$ , with no boundary effects included

$$\Omega_{free}(\mu = 0) \simeq -\frac{\pi^2}{45} V T^4 + \frac{1}{12} T^2 V [m^2 + (\xi - \frac{1}{6}) R] + \dots . \quad (6.25)$$

(It is possible to generalize this to include boundary effects and spacetimes whose dimension differs from 4 by using the results quoted in Sec. 4.2.)

We therefore find

$$\begin{aligned} \Omega(\mu = 0) &\simeq -\frac{\pi^2}{90} d_s V T^4 + \frac{1}{24} T^2 V \{ d_s [m^2 + (\xi - \frac{1}{6}) R \\ &\quad + V'_{tree}] + 2|\bar{\Phi}|^2 V''_{tree} \} + \dots \end{aligned} \quad (6.26)$$

as the leading terms in the high temperature expansion. For the  $U(1)$  model, which corresponds to  $d_s = 2$  (although there is only one complex scalar field the dimension refers to the real dimension), we recover the  $\mu \rightarrow 0$  result of (4.53). For the  $O(N)$  model it agrees with (3.24).

## 6.2 Inclusion of the chemical potential

In order to find the leading term at high temperature when the chemical potential is non-zero we will treat the terms in the action (6.7) which involve the chemical potential as an interaction and work to quadratic order in the chemical potential. It is easiest to use the imaginary time formalism in which (6.7) is replaced by

$$\begin{aligned} \bar{S} = & \int_0^\beta dt \int_\Sigma d\sigma_x \{ (\dot{\Phi}^\dagger + \mu^j \Phi^\dagger T_j)(\dot{\Phi} - \mu^j T_j \Phi) + |\nabla \Phi|^2 \\ & + (m^2 + \xi R) |\Phi|^2 + V_{tree} \} \end{aligned} \quad (6.27)$$

after the Wick rotation to imaginary time. We will define

$$\bar{S}_{int} = \int_0^\beta dt \int_\Sigma d\sigma_x \{ \mu^j (\Phi^\dagger T_j \dot{\Phi} - \dot{\Phi}^\dagger T_j \Phi) - \mu^j \mu^k \Phi^\dagger T_j T_k \Phi \} \quad (6.28)$$

as the interaction part of the action. We are therefore perturbing around the  $\mu = 0$  result for which we know the thermodynamic potential already from Sec. 7.1.

Up to second order in the interaction we have

$$\Gamma = \langle \bar{S}_{int} \rangle - \frac{1}{2} \langle \bar{S}_{int}^2 \rangle \quad (6.29)$$

where terms denoted by  $\langle \dots \rangle$  are to be Wick reduced with only one-particle irreducible graphs kept. (See Ref. [18] for example.) The basic result is

$$\langle \Phi_\alpha(x) \Phi_\beta^\dagger(x') \rangle = G_{\alpha\beta}(x, x') \quad (6.30)$$

where

$$\left( -\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 + \xi R \right) G_{\alpha\beta}(x, x') = \delta_{\alpha\beta} \delta(x, x') \quad (6.31)$$

defines the thermal Green function. (We have dropped the terms in  $V_{tree}$  here as the first order correction due to the self-interaction was considered earlier.)

We have

$$\langle \bar{S}_{int} \rangle = -\mu^j \mu^k (T_j T_k)_{\alpha\beta} \int_0^\beta dt \int_\Sigma d\sigma_x G_{\alpha\beta}(x, x). \quad (6.32)$$

Because we are only after the leading behaviour at high temperature, we may approximate  $G_{\alpha\beta}(x, x')$  by its flat spacetime expression. We have

$$G_{\alpha\beta}(x, x) \simeq \delta_{\alpha\beta} \sum_{n=-\infty}^{\infty} \frac{1}{\beta} \int \frac{d^D k}{(2\pi)^D} \left[ \left( \frac{2\pi n}{\beta} \right)^2 + k^2 + m^2 \right]^{-1}$$

$$\begin{aligned}
&= \delta_{\alpha\beta} T (4\pi)^{-D/2} (2\pi T)^{D-2} \Gamma(1 - D/2) \\
&\quad \times \left\{ \left( \frac{m}{2\pi T} \right)^{D-2} + 2 \sum_{n=1}^{\infty} \left[ n^2 + \left( \frac{m}{2\pi T} \right)^2 \right]^{D/2-1} \right\} . \tag{6.33}
\end{aligned}$$

For high  $T$ , taking  $D \rightarrow 3$  it is easily seen from (6.33) that

$$G_{\alpha\beta}(x, x) \simeq \frac{1}{12} T^2 \delta_{\alpha\beta} . \tag{6.34}$$

This leads to

$$\langle \bar{S}_{int} \rangle \simeq -\frac{1}{12} TV \mu^i \mu^j \text{tr}(T_i T_j) \tag{6.35}$$

as the leading contribution at high temperature.

The Wick reduction of the second term in (6.29), keeping only the part quadratic in the chemical potential, results in

$$\begin{aligned}
\langle \bar{S}_{int}^2 \rangle &= 4\mu^i \mu^j (T_i)_{\alpha\beta} (T_j)_{\gamma\delta} \int_0^\beta dt \int_0^\beta dt' \int_\Sigma d\sigma_x \int_\Sigma d\sigma_{x'} \\
&\quad \times \frac{\partial}{\partial t'} G_{\beta\gamma}(x, x') \frac{\partial}{\partial t} G_{\delta\alpha}(x', x) \tag{6.36}
\end{aligned}$$

after a short calculation. If we again use the flat spacetime Green function, with the understanding that we will only keep the leading order term, it may be shown from (6.36) that

$$\begin{aligned}
\langle \bar{S}_{int}^2 \rangle &= -8\mu^i \mu^j \text{tr}(T_i T_j) V (4\pi)^{-D/2} \Gamma(2 - D/2) \\
&\quad \times \sum_{n=1}^{\infty} (2\pi n T)^2 [(2\pi n T)^2 + m^2]^{D/2-2} . \tag{6.37}
\end{aligned}$$

Taking the high temperature limit and letting  $D \rightarrow 3$  results in

$$\langle \bar{S}_{int}^2 \rangle \simeq \frac{1}{6} TV \mu^i \mu^j \text{tr}(T_i T_j) . \tag{6.38}$$

Using (6.35) and (6.38) in (6.29) and noting that  $\Gamma = \beta\Omega$ , we have

$$\Omega \simeq -\frac{1}{6} T^2 V \mu^i \mu^j \text{tr}(T_i T_j) \tag{6.39}$$

as the leading order correction to the thermodynamic potential at high temperature when the chemical potentials are non-zero.

It is possible to relate  $\text{tr}(T_i T_j)$  to the quadratic Casimir invariant of the group using  $\text{tr}(T_i T_j) = (d_R C_2(G_R)/N) \delta_{ij}$ , where  $d_R$  is the real dimension of the representation  $G_R$  of the gauge group, and  $N$  is the dimension of the group. In the special case of  $O(N)$  the result in (6.39) reproduces (3.24) given earlier.



## 7 Bose-Einstein condensation

Let us finally come to the application of the calculations to the phenomenon of Bose-Einstein condensation for the case of a constant background field. The high-temperature effective action  $\Gamma[\bar{\Phi}]$  including the classical part of the background field  $\bar{\Phi}$  has the form

$$\begin{aligned} \frac{\Gamma[\bar{\Phi}]}{\beta V} &= (m^2 + \xi R)|\bar{\Phi}|^2 - \mu^i \bar{\Phi}^\dagger T_i \mu^j T_j \bar{\Phi} + V_{tree}(\bar{\Phi}) - \frac{\pi^2}{90} d_s T^4 \\ &+ \frac{1}{24} T^2 \left\{ d_s \left[ m^2 + \left( \xi - \frac{1}{6} \right) R + V'_{tree}(\bar{\Phi}) \right] + 2|\bar{\Phi}|^2 V''_{tree}(\bar{\Phi}) \right\} - \frac{1}{6} T^2 \mu^i \mu^j \text{Tr}(T_i T_j). \end{aligned} \quad (7.1)$$

Thus we find the equations of motion

$$\left[ m^2 + \xi R - \mu^i T_i \mu^j T_j + V'_{tree}(\bar{\Phi}) + \frac{T^2}{24} \left[ d_s V''_{tree}(\bar{\Phi}) + 2V''_{tree}(\bar{\Phi}) \right] \right] \bar{\Phi} = 0 \quad (7.2)$$

together with its conjugate. Hence the effective action has one minimum with unbroken symmetry,  $\bar{\Phi} = 0$ , and solution(s) with broken symmetry determined by

$$\det \left( m^2 + \xi R - \mu^i T_i \mu^j T_j + V'_{tree}(\bar{\Phi}) + \frac{T^2}{24} \left[ d_s V''_{tree}(\bar{\Phi}) + 2V''_{tree}(\bar{\Phi}) \right] \right) = 0. \quad (7.3)$$

For the  $\mathcal{O}(N)$ -model introduced in section 2 this is equivalent to

$$|\bar{\Phi}|^2 = \frac{3}{\lambda} \left[ \mu^\alpha \mu_\alpha - \left( m^2 + \xi R + \frac{N+2}{72} \lambda T^2 \right) \right]. \quad (7.4)$$

For  $N = 2$  this generalizes the flat space result [4, 5] and reduces to the result found in [15].

The discussion of eq. (7.4) then parallels completely the flat space discussion [2] and we will indicate it only briefly for  $N = 2$ . At very high temperature one finds the charge density

$$\rho = -\frac{1}{\beta} \frac{\partial \Gamma}{\partial \mu} = +\mu |\bar{\Phi}|^2 + \frac{1}{3} T^2 \mu + \dots, \quad (7.5)$$

the first piece corresponding to the charge  $Q_0$  in the ground state, the second one,  $Q_1$ , to the excited states. In the symmetric phase, fixing the charge  $Q$  and volume  $V$ , one thus have  $\mu = 3\rho/T^2$ . As temperature decreases,  $\mu$  will increase and will reach a value at which

$$\mu(T_c) = \frac{3\rho}{T_c^2} = \left( m^2 + \xi R + \frac{\lambda}{18} T_c^2 \right)^{\frac{1}{2}}. \quad (7.6)$$

Here  $T_c$  is, corresponding to (7.4), the critical temperature

$$T_c^2 = \frac{18}{\lambda} \left[ \mu^2(T_c) - m^2 - \xi R \right] \quad (7.7)$$

at which Bose-Einstein condensation and thus symmetry breaking occurs. For the charge in the ground state one then easily finds

$$Q_0 = Q \left[ 1 - \left( \frac{T}{T_c} \right)^2 \right], \quad (7.8)$$

which is identical to the form of the flat space result [2, 4, 5].

## 8 Conclusions

In this paper we continued the analysis started in the references [1-5] from flat to curved spacetime. Especially, we considered as a model a set of interacting scalar fields with a rigid gauge invariance under the action of an arbitrary gauge group  $G$ . In order to examine the phenomenon of Bose-Einstein condensation we derived the high temperature behaviour of the effective action of the theory. To this aim, different approaches were developed, one being based on the zeta function approach in combination with heat-kernel techniques, the second one being a weak coupling expansion.

As an application of our results, in the previous section we discussed the Bose-Einstein condensation for the case of a constant background field. Qualitatively the properties are the same as for flat Minkowski spacetime, however the results (7.6) and (7.7) show how the critical quantities at the condensation point depend on the curvature.

The results given in this article mainly include only the leading order correction to the thermodynamic potential at high temperature when the chemical potentials are non-zero. A more accurate result which includes sub-dominant terms in the temperature can be obtained by using the local momentum space expansion of Bunch and Parker [31] for the Green function in place of the flat spacetime Green function (6.33) we have used. It is also possible to extend the result to the next order in  $\mu$ . This will be given elsewhere.

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## A Appendix: Diagonalization of the matrix

The aim of this appendix is the diagonalization of the matrix  $U(\bar{\Phi})$ . The matrix  $S$  which diagonalizes the matrix  $U(\bar{\Phi})$  by  $U_{diag}(\bar{\Phi}) = S^{-1}U(\bar{\Phi})S$  is given by

$$S = \begin{pmatrix} e_1 & e_2 & e_3 & \cdot & \cdot & e_N \\ e_2 & -e_1 & 0 & \cdot & \cdot & 0 \\ e_3 & 0 & -e_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e_N & 0 & 0 & \cdot & \cdot & -e_1 \end{pmatrix}, \quad (\text{A.1})$$

with the inverse

$$S^{-1} = \begin{pmatrix} e_1 & e_2 & e_3 & \cdot & \cdot & e_N \\ e_2 & -\frac{1-e_2^2}{e_1} & \frac{e_3}{e_1}e_2 & \cdot & \cdot & \frac{e_N}{e_1}e_2 \\ e_3 & \frac{e_2}{e_1}e_3 & -\frac{1-e_3^2}{e_1} & \cdot & \cdot & \frac{e_N}{e_1}e_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e_N & \frac{e_2}{e_1}e_N & \frac{e_3}{e_1}e_N & \cdot & \cdot & -\frac{1-e_N^2}{e_1} \end{pmatrix}. \quad (\text{A.2})$$

With these results at hand, eqs. (3.12) and (3.18) for  $\zeta_z^0$  and  $\zeta_z^2$  are derived. The only thing to use furthermore is

$$\begin{aligned} tr \left\{ \exp \left[ -t \left( -U(\bar{\Phi}) - \frac{1}{6}R \right) \right] a_i \right\} &= tr \left\{ S^{-1} \exp \left[ -t \left( U(\bar{\Phi}) - \frac{1}{6}R \right) \right] S S^{-1} a_i S \right\} \\ &= tr \left\{ \begin{pmatrix} \exp(-tM_1^2) & 0 & \cdot & 0 \\ 0 & \exp(-tM_2^2) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \exp(-tM_2^2) \end{pmatrix} S^{-1} a_i S \right\}. \end{aligned} \quad (\text{A.3})$$

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